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LIMITING DISTRIBUTIONS OF MOMENT- AND QUANTILE-BASED MEASURES FOR SKEWNESS AND KURTOSIS

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Limiting distributions of moment- and quantile-based measures for skewness and kurtosis

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Abstract

Consider a random variable with distribution function F and central moments μ_i. Well-known measures for skewness and kurtosis of this distribution are given by the moment-based pair

$$(\beta_1, \beta_2) = (\mu_3/\mu_2^{3/2}, \mu_4/\mu_2^2)$$

Another pair of measures is based on the quantiles E_i defined by $F(E_i) = i/8$ (for continuous F). This quantile-based pair (S, T) is defined by

$$\begin{aligned} S &= (E_6 - 2E_4 + E_2)/(E_6 - E_2) \\ T &= (E_7 - E_5 + E_3 - E_1)/(E_6 - E_2) \end{aligned}$$

Let $(\underline{b}_1, \underline{b}_2)$ and $(\underline{s}, \underline{t})$ denote the natural estimators for (β_1, β_2) and (S, T), respectively, based on a random sample of size n. In this paper, for both pairs of random variables the limiting distributions for $n \to \infty$ are derived. The results are checked by means of simulations.

Two possible applications of these limiting distributions are discussed briefly: modelling and tests of normality. Here, modelling (or curve-fitting) means: describing a given empirical distribution by means of a simple theoretical probability distribution. This modelling can be based either on the pair $(\underline{b}_1, \underline{b}_2)$ or on the pair $(\underline{s}, \underline{t})$; the asymptotic efficiency of the two methods depends on the limiting distributions a.o. A test of normality based on $(\underline{b}_1, \underline{b}_2)$ is well-known; the pair $(\underline{s}, \underline{t})$ presents an alternative test statistic. The asymptotic power of both tests can be found from the limiting distributions.

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1 The limiting distribution of $(\underline{b}_1, \underline{b}_2)$

Consider a random variable $\underline{x}$ with moments $\alpha_i = E\{\underline{x}^i\}$; assume $\alpha_8 < \infty$. From

$$Cov(\underline{x}^i, \underline{x}^j) = \alpha_{i+j} - \alpha_i\alpha_j \quad (i,j = 1,2,3,4)$$

the covariance-matrix $\sum_4$ of $(\underline{x}, \underline{x}^2, \underline{x}^3, \underline{x}^4)$ follows:

$$\Sigma_4 = \begin{pmatrix} \alpha_2 - \alpha_1^2 & \alpha_3 - \alpha_1\alpha_2 & \alpha_4 - \alpha_1\alpha_3 & \alpha_5 - \alpha_1\alpha_4 \\ \alpha_3 - \alpha_1\alpha_2 & \alpha_4 - \alpha_2^2 & \alpha_5 - \alpha_2\alpha_3 & \alpha_6 - \alpha_2\alpha_4 \\ \alpha_4 - \alpha_1\alpha_3 & \alpha_5 - \alpha_2\alpha_3 & \alpha_6 - \alpha_3^2 & \alpha_7 - \alpha_3\alpha_4 \\ \alpha_5 - \alpha_1\alpha_4 & \alpha_6 - \alpha_2\alpha_4 & \alpha_7 - \alpha_3\alpha_4 & \alpha_8 - \alpha_4^2 \end{pmatrix} \tag{1.1}$$

For a random sample $(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)$ the sample moments $\underline{a}_i$ are defined by

$$\underline{a}_i = \frac{1}{n}\sum_{t=1}^{n} \underline{x}_t^i \quad (i = 1,2,3,4)$$

Then the multivariate version of the familiar Lindeberg-Lévy theorem gives

$$\sqrt{n}\begin{pmatrix} \underline{a}_1 - \alpha_1 \\ \underline{a}_2 - \alpha_2 \\ \underline{a}_3 - \alpha_3 \\ \underline{a}_4 - \alpha_4 \end{pmatrix} \xrightarrow{L} \underline{y} \sim N_4(0, \Sigma_4) \tag{1.2}$$

where $\xrightarrow{L}$ denotes convergence in distribution. See SERFLING (1980), p. 68.

To find the limiting distribution of the sample statistics

$$\begin{pmatrix} \underline{b}_1 \\ \\ \underline{b}_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{n}\sum_t(\underline{x}_t - \bar{\underline{x}})^3 / \left[\frac{1}{n}\sum_t(\underline{x}_t - \bar{\underline{x}})^2\right]^{3/2} \\ \\ \frac{1}{n}\sum_t(\underline{x}_t - \bar{\underline{x}})^4 / \left[\frac{1}{n}\sum_t(\underline{x}_t - \bar{\underline{x}})^2\right]^2 \end{pmatrix}$$

for skewness and kurtosis the following well-known result will be used.

Theorem 1. Let $\underline{y}_n, \underline{y}, c \in \mathbf{R}^k$ and $\varphi : \mathbf{R}^k \to \mathbf{R}^m$ with φ totally differentiable in c. If

$$\sqrt{n}(\underline{y}_n - c) \xrightarrow{L} \underline{y}$$

holds, then

$$\sqrt{n}[\varphi(\underline{y}_n) - \varphi(c)] \xrightarrow{L} \varphi'(c)\underline{y}$$

holds as well. □

Note that $\varphi'(c)$ is a $m \times k$-matrix of partial derivatives. Choose

$$\begin{aligned} \underline{y}_n &= (\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4)^\top \\ c &= (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^\top \end{aligned}$$

$$\varphi(y_1, y_2, y_3, y_4) = \begin{pmatrix} [y_3 - 3y_2y_1 + 2y_1^3]/[y_2 - y_1^2]^{3/2} \\ [y_4 - 4y_3y_1 + 6y_2y_1^2 - 3y_1^4]/[y_2 - y_1^2]^2 \end{pmatrix}$$

Then (1.2) may be rewritten as

$$\sqrt{n}(\underline{y}_n - c) \xrightarrow{L} \underline{y} \sim N_4(0, \textstyle\sum_4)$$

Since φ is totally differentiable at c and

$$\begin{aligned} \varphi(\underline{y}_n) &= \varphi(\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4) = (\underline{b}_1, \underline{b}_2)^\top \\ \varphi(c) &= \varphi(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\beta_1, \beta_2)^\top \end{aligned}$$

Theorem 1 implies

$$\sqrt{n}\begin{pmatrix} \underline{b}_1 - \beta_1 \\ \underline{b}_2 - \beta_2 \end{pmatrix} \xrightarrow{L} \varphi'(c)\underline{y} \sim N_2(0, \textstyle\sum^M) \tag{1.3}$$

where $\sum^M = \varphi'(c) \sum_4 [\varphi'(c)]^\top$; the superscript M stands for 'moments'.

The explicit calculation of $\sum^M$ can be simplified greatly by observing that this covariance-matrix is location-invariant. Hence, the calculation can be carried out for the special case $\mu_1 = 0$, implying $\alpha_i = \mu_i$ for all i. This gives

$$\Sigma^M = \varphi'(c)\Sigma_4[\varphi'(c)]^\top = AVA^\top$$

with

$$A = \begin{pmatrix} -3/\mu_2^{1/2} & -3\mu_3/2\mu_2^{5/2} & \mu_2 & 0 \\ -4\mu_3/\mu_2^2 & -2\mu_4/\mu_2^3 & 0 & 1/\mu_2^2 \end{pmatrix}$$

$$V = \begin{pmatrix} \mu_2 & \mu_3 & \mu_4 & \mu_5 \\ \mu_3 & \mu_4 - \mu_2^2 & \mu_5 - \mu_2\mu_3 & \mu_6 - \mu_2\mu_4 \\ \mu_4 & \mu_5 - \mu_2\mu_3 & \mu_6 - \mu_3^2 & \mu_7 - \mu_3\mu_4 \\ \mu_5 & \mu_6 - \mu_2\mu_4 & \mu_7 - \mu_3\mu_4 & \mu_8 - \mu_4^2 \end{pmatrix}$$

Then the elements $\sum_{ij}^M$ of the covariance-matrix $\sum^M$ are easily found to be

$$\begin{aligned}
\textstyle\sum_{11}^M &= \frac{\mu_6}{\mu_2^3} - 3\beta_1\frac{\mu_5}{\mu_2^{5/2}} + \beta_1^2(35+9\beta_2)/4 - 6\beta_2 + 9 \\
\textstyle\sum_{12}^M &= \textstyle\sum_{21}^M = \frac{\mu_7}{\mu_2^{7/2}} - \frac{3}{2}\beta_1\frac{\mu_6}{\mu_2^3} - (2\beta_2+3)\frac{\mu_5}{\mu_2^{5/2}} + 3\beta_1(\beta_2^2 + \beta_2/2 + 2\beta_1^2 + 4) \\
\textstyle\sum_{22}^M &= \frac{\mu_8}{\mu_2^4} - 4\beta_2\frac{\mu_6}{\mu_2^3} - 8\beta_1\frac{\mu_5}{\mu_2^{5/2}} + \beta_2^2(4\beta_2 - 1) + 16\beta_1^2(\beta_2+1)
\end{aligned}$$

The expressions for $\sum_{11}^M$ and $\sum_{22}^M$ can be found in STUART & ORD (1987), p. 344; the derivation was quite different, however. The formula for the limiting covariance of $\underline{b}_1$ and $\underline{b}_2$ seems to be new.

For the standardnormal and the standardexponential distribution $\sum^M$ was calculated directly as well, with the following results:

$$N(0,1) \Rightarrow \textstyle\sum^M = \begin{pmatrix} 6 & 0 \\ 0 & 24 \end{pmatrix}$$

$$Ne(1) \Rightarrow \textstyle\sum^M = \begin{pmatrix} 72 & 720 \\ 720 & 8064 \end{pmatrix}$$

These outcomes are in agreement with the general formulae.

2 The limiting distribution of $(\underline{s}, \underline{t})$

For any random variable $\underline{x}$ a p-quantile Q_p is defined by

$$P(\underline{x} < Q_p) \leq p, \quad P(\underline{x} \geq Q_p) \leq 1 - p$$

for $0 < p < 1$. If the distribution function F of $\underline{x}$ is differentiable in Q_p with $F'(Q_p) > 0$, this definition can be simplified to

$$F(Q_p) = p$$

Now, consider m given numbers p_i satisfying

$$0 < p_1 < p_2 < \ldots < p_m < 1$$

and denote the corresponding p_i-quantiles by Q_i. Let $\underline{q}_i$ denote the sample p_i-quantile $(i = 1, 2, \ldots, m)$ for a random sample $(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)$ from distribution function F. Then the following limit theorem holds; see SERFLING (1980), p. 80.

Theorem 2. If F is differentiable in Q_i with derivative $f(Q_i) = F'(Q_i) > 0$ $(i = 1, 2, \ldots, m)$, then

$$\sqrt{n} \begin{pmatrix} \underline{q}_1 - Q_1 \\ \underline{q}_2 - Q_2 \\ \vdots \\ \underline{q}_m - Q_m \end{pmatrix} \xrightarrow{L} \underline{y} \sim N_m(0, \Sigma)$$

where $\sum_{ij} = p_i(1 - p_j)/[f(Q_i)f(Q_j)]$ for $i \leq j$ and $\sum_{ij} = \sum_{ji}$. $\square$

The seven quantiles corresponding with $p_i = i/8$ are called octiles and will be denoted by E_i, $i = 1, 2, \ldots, 7$. Alternative measures for skewness and kurtosis exist, based on these octiles:

$$\begin{cases} S = (E_6 - 2E_4 + E_2)/(E_6 - E_2) \\ T = (E_7 - E_5 + E_3 - E_1)/(E_6 - E_2) \end{cases} \tag{2.1}$$

Of course, the quantile-measure S is the well-known quantile-coefficient of skewness; for the properties of T see MOORS (1988) or WAGEMAKERS et al. (1992).

Now the limiting distribution of the seven sample octiles $\underline{e}_i$ is an immediate consequence of Theorem 2.

Corollary 1. If F is differentiable in E_i with derivative $f(E_i) = F'(E_i) > 0$ $(i = 1, 2, \ldots, 7)$, then

$$\sqrt{n}\begin{pmatrix} \underline{e}_1 - E_1 \\ \underline{e}_2 - E_2 \\ \vdots \\ \underline{e}_7 - E_7 \end{pmatrix} \xrightarrow{L} \underline{y} \sim N_7(0, \Sigma_7)$$

where

$$\Sigma_7 = \frac{1}{64}\begin{pmatrix} 7c_1^2 & 6c_1c_2 & 5c_1c_3 & 4c_1c_4 & 3c_1c_5 & 2c_1c_6 & c_1c_7 \\ 6c_1c_2 & 12c_2^2 & 10c_2c_3 & 8c_2c_4 & 6c_2c_5 & 4c_2c_6 & 2c_2c_7 \\ 5c_1c_3 & 10c_2c_3 & 15c_3^2 & 12c_3c_4 & 9c_3c_5 & 6c_3c_6 & 3c_3c_7 \\ 4c_1c_4 & 8c_2c_4 & 12c_3c_4 & 16c_4^2 & 12c_4c_5 & 8c_4c_6 & 4c_4c_7 \\ 3c_1c_5 & 6c_2c_5 & 9c_3c_5 & 12c_4c_5 & 15c_5^2 & 10c_5c_6 & 5c_5c_7 \\ 2c_1c_6 & 4c_2c_6 & 6c_3c_6 & 8c_4c_6 & 10c_5c_6 & 12c_6^2 & 6c_6c_7 \\ c_1c_7 & 2c_2c_7 & 3c_3c_7 & 4c_4c_7 & 5c_5c_7 & 6c_6c_7 & 7c_7^2 \end{pmatrix}$$

with $c_i = 1/f(E_i)$. $\square$

The natural estimators for S and T are the sample statistics

$$\begin{cases} \underline{s} = (\underline{e}_6 - 2\underline{e}_4 + \underline{e}_2)/(\underline{e}_6 - \underline{e}_2) \\ \underline{t} = (\underline{e}_7 - \underline{e}_5 + \underline{e}_3 - \underline{e}_1)/(\underline{e}_6 - \underline{e}_2) \end{cases} \tag{2.2}$$

Their limiting distribution can be found with the help of Theorem 1. By taking

$$\begin{aligned} \underline{y}_n &= (\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_7)^\top \\ c &= (E_1, E_2, \ldots, E_7)^\top \end{aligned}$$

$$\varphi(y_1, y_2, \ldots, y_7) = \begin{pmatrix} (y_6 - 2y_4 + y_2)/(y_6 - y_2) \\ (y_7 - y_5 + y_3 - y_1)/(y_6 - y_2) \end{pmatrix}$$

the result of Corollary 1 can be written as

$$\sqrt{n}(\underline{y}_n - c) \xrightarrow{L} \underline{y} \sim N_7(0, \textstyle\sum_7)$$

Then Theorem 1 gives

$$\sqrt{n} \begin{pmatrix} \underline{s} - S \\ \underline{t} - T \end{pmatrix} \xrightarrow{L} \varphi'(c)\underline{y} \sim N_2(0, \textstyle\sum^Q) \tag{2.3}$$

where $\sum^Q = \varphi'(c) \sum_7 [\varphi'(c)]^{\top}$; the Q stands for 'quantiles'. With the notation

$$D_6 = E_6 - E_4, \; D_4 = E_4 - E_2, \; D = E_6 - E_2$$

the elements $\sum_{ij}^Q$ of $\sum^Q$ are found to be given by

$$\begin{aligned}
4D^4 \textstyle\sum_{11}^Q &= (c_2 D_6 + c_6 D_4)^2 + 2(c_2 D_6 - c_4 D)^2 + 2(c_6 D_4 - c_4 D)^2 \\
16D^3 \textstyle\sum_{12}^Q &= c_2 D_6(-3c_1 + 6c_2 T + 5c_3 - 3c_5 - 2c_6 T + c_7) + \\
&\quad + c_4(2c_1 - 4c_2 T - 6c_3 + 6c_5 + 4c_6 T - 2c_7) + \\
&\quad + c_6 D_4(-c_1 + 2c_2 T + 3c_3 - 5c_5 - 6c_6 T + 3c_7)
\end{aligned}$$

$$\textstyle\sum_{21}^Q = \sum_{12}^Q$$

$$\begin{aligned}
64D^2 \textstyle\sum_{22}^Q &= 6(c_1 - c_2 T)^2 + 5(c_1 - c_3)^2 - 3(c_1 - c_5)^2 - 2(c_1 - c_6 T)^2 + \\
&\quad + (c_1 - c_7)^2 + 10(c_2 T + c_3)^2 - 6(c_2 T + c_5)^2 + 4(c_2 T - c_6 T)^2 + \\
&\quad - 2(c_2 T - c_7)^2 + 9(c_3 - c_5)^2 - 6(c_3 + c_6 T)^2 - 3(c_3 - c_7)^2 + \\
&\quad + 10(c_5 + c_6 T)^2 + 5(c_5 - c_7)^2 + 6(c_6 T - c_7)^2
\end{aligned}$$

The results for the two special distributions are:

$$N(0,1) \Rightarrow \textstyle\sum^Q = \begin{pmatrix} 1.839 & 0 \\ 0 & 3.153 \end{pmatrix}$$

$$Ne(1) \Rightarrow \textstyle\sum^Q = \begin{pmatrix} 1.782 & -0.151 \\ -0.151 & 5.104 \end{pmatrix}$$

3 Simulations

As a final check of the formulae for the covariance-matrices derived before, as well as to investigate how accurate the limiting distributions are for finite n, rather extensive simulation studies were undertaken. The core of these studies consisted of the following four steps.

1. Choose a certain probability distribution and calculate the pairs of measures (β_1, β_2) and (S, T).
2. Simulate from this distribution a random sample of size n and calculate the pairs of statistics (b_1, b_2) and (s, t).
3. Repeat step 2 k times, independently.
4. Calculate the means and the covariance-matrix for the k pairs $\sqrt{n}(b_1 - \beta_1, b_2 - \beta_2)$, as well as for the k pairs $\sqrt{n}(s - S, t - T)$.

In this procedure, the following choices were made.

(i) Table 1 shows the distributions used in step 1.

Table 1. Distributions used in simulation studies.

type	parameters (p, q)			
$N(\mu, \sigma^2)$	(0,1)			
$B(p, q)$	(0.5,1)	(1,2)	(1,4)	(2,4)
$B2(p, q)$	(3,15)	(4,28)	(5,15)	
$\Gamma(p, q)$	(1,1)	(1,5)		
$I\Gamma(p, q)$	(1,9)	(1,12)		

Here, $B(p, q)$ indicates the beta-distribution with density

$$f(x) = x^{p-1}(1-x)^{q-1}/B(p,q), \qquad 0 < x < 1; \quad p, q > 0$$

while $B2(p, q)$ is the beta-distribution of the second kind with density

$$f(x) = x^{p-1}/(1+x)^{p+q}/B(p,q), \quad x > 0; \quad p, q > 0$$

$\Gamma(p,q)$ denotes the gamma-distribution with density

$$f(x) = p^q x^{q-1} e^{-px}/\Gamma(q), \qquad x > 0; \quad p,q > 0$$

and $I\Gamma(p,q)$ is the inverse gamma-distribution with density

$$f(x) = p^q x^{-q-1} e^{-p/x}/\Gamma(q) \qquad x > 0; \quad p,q > 0$$

Note that all distributions in Table 1 belong to the Pearson-system of distributions; the reason for this is given in Section 4. Table 2 presents the values for the pairs (β_1, β_2) and (S,T) for the distributions selected in Table 1.

Table 2. Moments- and quantile-based measures for skewness and kurtosis.

Distributions	β_1	β_2	S	T
$N(0,1)$	0	3	0	1.233
$B(0.5,1)$	0.639	2.143	0.25	1
$B(1,2)$	0.566	2.4	0.132	1.103
$B(1,4)$	1.050	3.696	0.197	1.190
$B(2,4)$	0.468	2.625	0.078	1.167
$B2(3,15)$	1.683	8.180	0.171	1.289
$B2(4,28)$	1.282	5.819	0.139	1.266
$B2(5,15)$	1.480	7.128	0.147	1.281
$\Gamma(1,1)$	2	9	0.262	1.306
$\Gamma(1,5)$	0.894	4.2	0.104	1.243
$I\Gamma(1,9)$	1.764	9.8	0.151	1.302
$I\Gamma(1,12)$	1.405	7.083	0.131	1.284

(If less than three decimals are shown, the value is exact.)

(ii) In step 2 the following sample sizes n were considered:

50, 100, 200, 2000

(ii) In step 3 two values were taken for the number of iterations k:

20, 200

Tables 3 and 4 show the detailed results for the standard normal distribution; the last lines for $n = \infty$ are obtained from the theoretical results $\sum^M$ and $\sum^Q$ in Section 1 and 2, respectively.

Table 3. Simulations and limiting values; $(\underset{\sim}{b}_1, \underset{\sim}{b}_2)$, $N(0,1)$.

k	n	$\sqrt{n}E(\underset{\sim}{b}_1 - \beta_1)$	$\sqrt{n}E(\underset{\sim}{b}_2 - \beta_2)$	$nV(\underset{\sim}{b}_1)$	$nV(\underset{\sim}{b}_2)$	$n\ Cov\ (\underset{\sim}{b}_1, \underset{\sim}{b}_2)$
20	50	0.059	0.939	6.527	41.716	9.436
	100	−0.714	−1.596	5.067	19.704	−3.059
	200	−0.295	−0.968	4.889	33.844	−5.389
	2000	0.179	0.421	4.121	18.570	−1.890
200	50	−0.240	−0.893	5.068	15.099	−0.068
	100	0.095	−0.840	4.683	17.781	−0.035
	200	−0.013	−0.461	5.453	21.181	1.185
	2000	0.296	0.570	5.180	23.137	0.283
	∞	0	0	6	24	0

Table 4. Simulations and limiting values; $(\underset{\sim}{s}, \underset{\sim}{t})$, $N(0,1)$.

k	n	$\sqrt{n}E(\underset{\sim}{s} - S)$	$\sqrt{n}E(\underset{\sim}{t} - T)$	$nV(\underset{\sim}{s})$	$nV(\underset{\sim}{t})$	$n\ Cov\ (\underset{\sim}{s}, \underset{\sim}{t})$
20	50	−0.395	0.017	1.556	2.538	0.146
	100	0.548	0.391	1.757	4.341	0.402
	200	0.393	0.159	0.964	1.259	0.010
	2000	0.845	−0.170	1,532	1.969	0.330
200	50	−0.224	−0.149	1.887	3.940	0.256
	100	0.103	0.395	1.745	3.378	−0.034
	200	−0.039	0.395	1.931	3.487	−0.026
	2000	−0.048	0.654	1.524	3.815	−0.283
	∞	0	0	1.839	3.153	0

As was to be expected, the simulated outcomes are closer to the limiting values when the sample size n and/or the number of iterations k increase.

To save space, only the simulated outcomes for the values $k = 200$ and $n = 2000$ are reported here (Tables 5 and 6) for the other distributions of Table 1. Fuller details can be found in COENEN (1992).

Table 5. Simulations for $k = 200$ and $n = 2000$ ('sim') and limiting values ('lim'); $(\underline{b}_1, \underline{b}_2)$.

Distribution		$\sqrt{n}E(\underline{b}_1 - \beta_1)$	$\sqrt{n}E(\underline{b}_2 - \beta_2)$	$nV(\underline{b}_1)$	$nV(\underline{b}_2)$	$n\ Cov(\underline{b}_1, \underline{b}_2)$
$B(0.5, 1)$	sim	0.034	0.137	2.752	8.582	4.429
	lim	0	0	2.735	8.144	4.313
$B(1, 2)$	sim	0.021	0.051	2.601	9.418	4.225
	lim	0	0	2.499	8.887	3.927
$B(1, 4)$	sim	−0.186	−0.526	4.492	57.668	14.709
	lim	0	0	5.125	67.962	17.213
$B(2, 4)$	sim	−0.072	−0.079	2.896	12.315	4.667
	lim	0	0	2.999	12.979	4.789
$B2(3, 15)$	sim	0.288	6.858	139.41	24811	1750
	lim	0	0	119.35	22747	1488
$B2(4, 28)$	sim	0.364	4.515	49.038	3200	369.44
	lim	0	0	39.885	2944	315.03
$B2(5, 15)$	sim	−0.260	−2.777	81.860	8564	793.32
	lim	0	0	88.267	13493	982.04
$\Gamma(1, 1)$	sim	−0.274	−2.702	63.389	6481	615.68
	lim	0	0	72	8064	720
$\Gamma(1, 5)$	sim	0.103	−0.012	12.916	330.18	59.739
	lim	0	0	14.400	410.11	69.551
$I\Gamma(1, 9)$	sim	−0.221	−4.742	327.13	79197	4864
	lim	0	0	386.84	460077	9488
$I\Gamma(1, 12)$	sim	−0.659	−7.124	104.62	1891	1052
	lim	0	0	109.74	23002	1381

Table 6. Simulations for $k = 200$ and $n = 2000$ ('sim') and limiting values ('lim'); $(\underset{\sim}{s}, \underset{\sim}{t})$.

Distribution		$\sqrt{n}E(\underset{\sim}{s} - S)$	$\sqrt{n}E(\underset{\sim}{t} - T)$	$nV(\underset{\sim}{s})$	$nV(\underset{\sim}{t})$	$n\ Cov\ (\underset{\sim}{s}, \underset{\sim}{t})$
$B(0.5, 1)$	sim	−0.022	−0.015	1.956	2.662	0.061
	lim	0	0	2.092	2.623	0.094
$B(1, 2)$	sim	−0.083	−0.054	1.774	2.733	−0.135
	lim	0	0	1.900	2.648	−0.049
$B(1, 4)$	sim	0.097	0.020	1.802	3.744	−0.261
	lim	0	0	1.725	3.086	−0.136
$B(2, 4)$	sim	−0.058	0.101	1.938	3.190	0.228
	lim	0	0	1.854	2.847	−0.045
$B2(3, 15)$	sim	0.106	0.109	1.646	4.562	0.107
	lim	0	0	1.806	4.198	−0.115
$B2(4, 28)$	sim	−0.035	−0.015	1.589	4.343	−0.193
	lim	0	0	1.817	3.796	−0.093
$B2(5, 15)$	sim	0.038	0.316	1.580	3.424	−0.215
	lim	0	0	1.817	3.986	−0.101
$\Gamma(1, 1)$	sim	−0.050	−0.061	1.874	5.098	−0.340
	lim	0	0	1.783	5.104	−0.151
$\Gamma(1, 5)$	sim	0.070	0.099	1.967	3.393	0.096
	lim	0	0	1.830	3.445	−0.070
$I\Gamma(1, 9)$	sim	0.012	−0.161	1.663	4.236	−0.209
	lim	0	0	1.820	4.230	−0.110
$I\Gamma(1, 12)$	sim	0.088	0.030	1.408	4.307	−0.187
	lim	0	0	1.819	3.913	−0.091

The overall agreement is good, with the possible exception of the $V(\underset{\sim}{b}_2)$-values for the inverse-gamma-distributions. Note however, that these distributions have very high β_2-values; probably, the number of replications was too small to obtain a good agreement.

4 Application in modelling

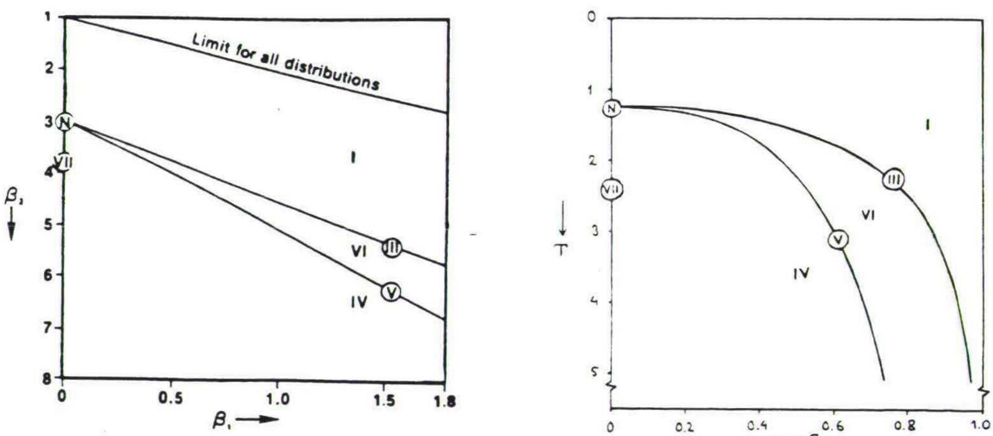
Modelling an empirical frequency distribution by means of a simple theoretical probability distribution is an interesting issue in applied statistics. One of the advantages is that the analytical expression for the density may lead to explicit cost formulae, e.g. in inventory problems. A reasonable approach to this modelling problem is to demand that measures for location, dispersion, skewness and kurtosis for the two distributions coincide. Traditionally, the moment-based quartet of measures $(\mu, \mu_2, \beta_1, \beta_2)$ was used hereby. An alternative is the quantile-based quartet (Q, R, S, T), with the median Q and the half interquantile range R defined as

$$Q = E_4, \quad R = (E_6 - E_2)/2$$

So, this last quartet can be found from the seven octiles.

Of course, the values of such a foursome of measures do not uniquely determine a probability distribution in general. They do, however, within specific systems of distributions like Pearson's or Johnson's. Both systems have the property that any (location-scale-free) distribution within the system has a different pair of values for the measures (β_1, β_2). Similarly, their is a one-one-correspondence between these (location-scale-free) distributions and the pairs (S, T). Figure 1 shows these one-one-relations for the Pearson system; the (symmetric) half-planes with negative β_1- and S-values have been omitted.

Figure 1. (β_1, β_2)- and (S, T)-plane for the Pearson system.



The Roman capitals indicate different types of distributions; e.g. I corresponds with the beta-distributions and III with the gamma distributions. N is the class of all normal distributions. See WAGEMAKERS (1992) for details.

Now, if for an empirical frequency distribution the values of the four moment-measures have been calculated, there is (at most) one distribution within the Pearson system with the same values for the quartet $(\mu, \mu_2, \beta_1, \beta_2)$. This unique distribution can be taken as a model for the empirical one. In the same way, the (unique) distribution within Pearson's system, having the same values of the quantile-based measures (Q, R, S, T) as the empirical distribution, presents another model.

The important question now is which of these two models is to be preferred: does the (classical) approach using $(\mu, \mu_2, \beta_1, \beta_2)$ lead to a better-fitting model than the quantile-based foursome (Q, R, S, T) does? The limiting distributions derived in this paper present a first step in answering this question: comparison of Tables 5 and 6 immediately shows that – for n large – $(\underline{s}, \underline{t})$ has (much) smaller variances than $(\underline{b}_1, \underline{b}_2)$. This implies that with the same size (S, T) can be estimated (much) more accurately than (β_1, β_2).

However, comparing the accuracy of the two modelling methods is more complicated. Let $D(p, q)$ be one of the distributions in Pearson's system, D indicating the type; a random sample is drawn, leading to the pairs $(\underline{b}_1, \underline{b}_2)$ and $(\underline{s}, \underline{t})$. They lead to unique distributions, say

$$\underline{D}_M(\underline{p}_M, \underline{q}_M), \quad \underline{D}_Q(\underline{p}_Q, \underline{q}_Q)$$

respectively. The random variables $\underline{p}_M, \underline{p}_Q$ and $\underline{q}_M, \underline{q}_Q$ can be seen as estimators for p and q, respectively. Note, however, that the types $\underline{D}_M$ and $\underline{D}_Q$ are not necessarily identical to the sampled type D!

If $\underline{D}_M = \underline{D}_Q = D$ with probability 1, the accuracy of the modelling processes would only depend on the efficiency of the estimators $(\underline{p}_M, \underline{q}_M)$ and $(\underline{p}_Q, \underline{q}_Q)$, hence on the matrices of mean squares errors, S_M and S_Q, say. For the moment method, S_M is defined as

$$S_M = E[(\underline{p}_M - p, \underline{q}_M - q)(\underline{p}_M - p, \underline{q}_M - q)^\top]$$

while a similar expression holds for S_Q. This method breaks down however, since in practice $P(\underline{D}_M = \underline{D}_Q = D_Q) = 1$ will never hold.

Some small-scale experiments were performed on this subject. From $D(p,q) = B(0.5,1)$ twenty random samples of size $n = 50$ were drawn. In all cases but one the sample pairs (b_1, b_2) and (s,t) lead back to beta distributions – hence $D_M = D_Q = D$, the exception being a sample with $(s,t) = (0.241, 1.571)$, corresponding with a Pearson distribution of type IV. From the nineteen remaining samples we calculated the estimates $\hat{S}_M$ and $\hat{S}_Q$, where e.g.

$$\hat{S}_M = \frac{1}{19}\sum_{i=1}^{19}(p_{Mi} - 0.5, q_{Mi} - 1)(p_{Mi} - 0.5, q_{Mi} - 1)^{\top}$$

and i denotes the different sample. The results were

$$\hat{S}_M = \begin{pmatrix} 0.0296 & 0.0176 \\ 0.0176 & 0.0520 \end{pmatrix}, \quad \hat{S}_Q = \begin{pmatrix} 0.0202 & 0.0337 \\ 0.0337 & 0.1090 \end{pmatrix}$$

Since $\hat{S}_M - \hat{S}_Q$ is indefinite, no conclusion can be drawn about which modelling process is better - even if the problem of identifying the right **type** of distribution is neglected.

Better measures for the accuracy of the modelling methods can be obtained by comparing the distribution functions of the model and the sampled distributions. Denoting these by F_M (for the moment-based model) and F, respectively, useful measures of fit are the Kolmogorov-Smirnov-distance

$$\max_x |F_M(x) - F(x)|$$

or the Cramer-Von Mises statistic

$$\int_0^1 [F_M(x) - F(x)]^2 dF(x)$$

We plan to do some future research in this direction.

5 Application in test for normality

Their exist quite a few tests for normality; SHAPIRO et al. (1968) discuss and compare nine. Among them are two tests using as test statistic $\underline{b}_1$ – in our notation – and $\underline{b}_2$ separately. These test statistics can be used in combination as well. D'AGOSTINO et al.

(1973) were the first to discuss a test based on the simultaneous distribution of $(\underline{b}_1, \underline{b}_2)$. The original testing problem:

H_o : the sample stems from some normal distribution,
H_a : the sample stems from any other distribution within the Pearson system

is in fact replaced by

$$H'_o : (\beta_1, \beta_2) = (0,3), \quad H'_a : (\beta_1, \beta_2) \neq (0,3)$$

The limiting distribution of $(\underline{b}_1, \underline{b}_2)$ obtained in Section 1 implies that for large n

$$\underline{b}_1 \approx N(0, 6/n), \quad \underline{b}_2 \approx N(3, 24/n), \quad \underline{b}_1 \text{ and } \underline{b}_2 \text{ independent}$$

holds under the null hypotheses. Here $\approx$ means 'approximately distributed as'. Hence, under H_o

$$\underline{b} = n[\underline{b}_1^2/6 + (\underline{b}_2 - 3)^2/24] \approx \chi_2^2 \tag{5.1}$$

holds, so that $\underline{b}$ can be used as test statistic, the critical region being given by

$$\{b : b > \chi^2_{2;\alpha}\}$$

For large n, this test has a size of approximately α.

A problem with this test is the very slow rate of convergence of $(\underline{b}_1, \underline{b}_2)$ to the binormal limiting distribution of Section 1. So, for moderate sample sizes this test is rather inaccurate. Especially the distribution of $\underline{b}_2$ is highly nonnormal, even if the sampled distribution is normal and even for large sample sizes. BOWMAN & SHENTON (1975) refined the test and obtained test contours for moderate n, based on a better approximation of the simultaneous distribution of $(\underline{b}_1, \underline{b}_2)$.

An alternative test may be based on $(\underline{s}, \underline{t})$. For large n, Section 2 implies

$$\underline{s} \approx N(0, 1.839), \quad \underline{t} \approx N(1.233, 3.153), \quad \underline{s} \text{ and } \underline{t} \text{ independent}$$

under H_o and, consequently,

$$\underline{u} = n[\underline{s}^2/1.839 + (\underline{t} - 1.233)^2/3.153] \approx \chi_2^2 \tag{5.2}$$

The critical region for test statistic $\underline{u}$ equals

$$\{u : u > \chi^2_{2;\alpha}\}$$

For large n, this test has a size of approximately α as well.

Since $\underline{t}$ is much less influenced by outliers in the sample, it is to be expected that the convergence in distribution to the normal is much faster. This means that the approximation (5.2) is (much) better than (5.1) for moderate sample sizes. Therefore, this alternative quantile-based test may be a serious competitor to the existing ones. Approximate power-calculations can be made with the help of the general limiting distributions in Section 2. This topic too will be further investigated in the (near) future.

Note that there is nothing special in the null-hypothesis of normality: for **any** distribution within the Pearson system a similar test can be derived.

6 Discussion

This paper concentrated on measures for skewness and kurtosis only. Of course, limiting distributions for the quartets $(\underline{\bar{x}}, \underline{s}^2, \underline{b}_1, \underline{b}_2)$ and $(\underline{q}, \underline{r}, \underline{s}, \underline{t})$ can be derived in a similar fashion.

In Section 4 attention was mainly directed towards Pearson's system of distributions. Similar results can be derived for Johnson's system, since there too all distributions have distinct pairs of values (β_1, β_2) and (S, T). Still other systems may be considered as classes of possible models; interesting candidates are the SCHMEISER-DEUTSCH (1978) system and Burr's system, c.f. STUART & ORD (1987), p. 242.

Apart from the methods discussed in this paper - modelling by equating measures for location, dispersion, skewness and kurtosis - quite different methods of modelling can be used. We mention two possibilities: maximum likelihood and kernel estimation.

(i) Maximum likelihood (ML) estimation is usually applied when the type of density is given and only some parameters are unknown. ML-estimators have, under rather mild conditions, nice asymptotic properties, like consistency and efficiency. In our case, a number of different types of densities are taken into consideration simultaneously. Since, however, Pearson's system is based upon a single differential equation with a limited number of parameters, ML-estimation may be possible in this case.

(ii) kernel estimation comes down to smoothing the empirical frequency distribution. From the sample $(\underline{x}_1, \underline{x}_2, ..., \underline{x}_n)$, the density f is estimated by $\underline{f}$, where

$$\underline{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} K[(x - \underline{x}_i)/h]$$

for all x. The kernel function K is concentrated around 0 and satisfies

$$\int_{-\infty}^{\infty} K(x)dx = 1$$

The bandwidth h determines the level of smoothing. Note that the resulting estimated density can not be described by a simple analytical formula.

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